Optimal Designs Approach to Portfolio Selection

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In order to obtain the best tradeoff between risk and return, optimization algorithms are particularly useful in asset allocation in a portfolio mix. Such algorithms and proper solution techniques are very essential to investors in order to circumvent distress in business outfits. In this paper, we show that by minimizing the total variance of the portfolio involving stocks in two Nigerian banks which is a measure of risk, optimal allocation of investible funds to the portfolio mix is obtained. A completely new solution technique – modified super convergent line series algorithm which makes use of the principles of optimal designs of experiment is used to obtain the desired optimizer.

Keywords: Portfolio selection, minimum variance, optimal designs, optimal allocation.

1. Introduction

In every investment, there is a tradeoff between risk and returns on such investment. An investor therefore must be willing to take on extra risk if he intends to obtain additional expected returns. However, there must be a balance between risk and returns that suits individual investors, Neveu (1985).

Great care must be taken by any investor in the allocation of his investible funds to a list of investments open to him in order to minimize the total risk involved. A mathematical model to suit a problem of this nature and in particular, a quadratic programming model for portfolio selection was developed by Markowitz (1952, 1959).

A portfolio mix is a set of investments that an investor can invest in while a portfolio risk refers to the risk common to all securities in the portfolio mix and this is equated with the standard deviation of returns, Ebrahim (2008).

The purpose of the investment of cash in portfolios of securities is to provide a better return than would be earned if the money were retained as cash or as a bank deposit. The return may come in the form of a regular income by way of dividends or interest or by way of growth in capital value or by a combination of both regular income and growth in capital value, Cohen and Zinbarg (1967). Thus, the real objective of portfolio construction becomes that of achieving the maximum return with minimum risk, Weaver (1983).

Grubel (1968) showed that higher returns and lower risks than the usual are obtained from international diversification. Arnott and Copeland (1985) have also shown that the business cycle has a significant effect on security returns. On their part, Chen, Roll and Ross (1986) determined that certain macroeconomic variables are significant indicators of changes in stock returns. Contributing further, Bauman and Miller (1995) showed that the evaluation of portfolio performance should take place through a complete stock market cycle because of differences in performance during the market cycle. Macedo (1995) demonstrates that switching between relative strength and relative value strategies can increase returns in an international portfolio.

Since portfolio selection problem is a quadratic programming problem which involves a minimization of risk associated with such investment by minimizing the total variance which is a measure of the risk involved, Francis (1980), suitable solution technique should be adopted to obtain optimal solution. Etukudo and Umoren (2009) have shown that it is easier and in fact better to use modified super convergent line series algorithm ($MSCLS_Q$) which uses the principles of optimal designs of experiment in solving quadratic programming problems rather than using the traditional solution technique of modified simplex method. This paper therefore focuses on optimal designs approach to optimal allocation of investible funds in a portfolio mix.

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2. A quadratic programming model for portfolio selection

For a quadratic programming model for portfolio selection, let

n = number of stocks to be included in the portfolio

x_j = number of shares to be purchased in stocks j, j = 1, 2, …, n

Y_j = returns per unit of money invested in stocks j at maturity

Assuming the values of Y_j are random variables, then

$$E(Y_j) = \overline{Y}_j; \quad j = 1,2,\dots,n \tag{1}$$

$$V = \sigma_{ij} = E\left[\left(Y_i - \overline{Y}_i\right)\left(Y_j - \overline{Y}_j\right)\right] \tag{2}$$

where $E(Y_j)$ is the mathematical expectation of Y_j and V is the variance – covariance matrix of the returns. See Gruyter (1987), Parsons (1977) and Etukudo et al (2009). Hence, the variance of the total returns or the portfolio variance is given by

$$f(\mathbf{x}) = \mathbf{X}'\mathbf{V}\mathbf{X} = \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j \tag{3}$$

which measures the risk of the portfolio selected. The non-negativity constraints are

$$x_j \geq 0, \; j = 1, 2, \dots, n \tag{4}$$

Assuming the minimum expected returns per unit of money invested in the portfolio is B, then

$$\sum_{j=1}^{n}\overline{Y}_j x_j \geq B \tag{5}$$

2.1 Minimization of the total risk involved in the portfolio

By minimizing the total variance, f(**x**) of the portfolio, the total risk involved in the portfolio is minimized. In order to obtain a minimum point of equation 3, f(**x**) must be a convex function, Hillier and Lieberman (2006). That is,

$$\frac{\partial^2 f(x_j)}{\partial x_1^2}\frac{\partial^2 f(x_j)}{\partial x_2^2}\cdots\frac{\partial^2 f(x_j)}{\partial x_j^2} - \left[\frac{\partial^2 f(x_j)}{\partial x_i \partial x_j}\right]^2 - \dots - \left[\frac{\partial^2 f(x_j)}{\partial x_i \partial x_j}\right]^2 \geq 0 \tag{6}$$

where

$$\begin{aligned} &\frac{\partial^2 f(x_j)}{\partial x_1^2} \geq 0 \\ &\quad\vdots \\ &\frac{\partial^2 f(x_j)}{\partial x_j^2} \geq 0 \end{aligned} \tag{7}$$

where $i \neq j = 1, 2, \dots, n$. Strict inequalities of 6 and 7 imply that f(**x**) is strictly convex and hence, has a global minimum at **x***. From equation 3 and inequalities 4 and 5, the portfolio selection model is given by;

$$\text{Min } f(x) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j$$

subject to:

$$\sum_{j=1}^{n}\overline{Y}_j x_j \geq B \qquad ; \quad x_j \geq 0, \; j = 1, 2, \dots, n$$

Remark

The expected values, $\overline{Y}_j$ and the variance – covariance matrix, σ_{ij} are based on data from historical records.

3. Modified super convergent line series algorithm ($MSCLS_Q$), Umoren and Etukudo (2009)

The sequential steps involved in $MSCLS_Q$ are given as follows:

Step 1: Let the response surface be

$$y = c_0 + c_1x_1 + c_2x_2 + q_1x_1^2 + q_2x_1x_2 + q_3x_2^2 \qquad x_1, x_2 \in G_i,\ i = 1, 2, \ldots, k^*$$

Select N support points such that $3k^* \le N \le 4k^*$ where $2 \le k^* \le 3$ is the number of partitioned groups desired. By arbitrarily choosing the support points as long as they do not violate any of the constraints, make up the initial design matrix

$$X = \begin{bmatrix} 1 & x_{11} & x_{21} \\ 1 & x_{12} & x_{22} \\ \vdots & \vdots & \vdots \\ 1 & x_{1N} & x_{2N} \end{bmatrix}$$

Step 2: Partition X into k* groups with equal number of support points and obtain the design matrix, X_i, i = 1, 2, …, k* for each group. Obtain the information matrices $M_i = X_i'X_i$, i = 1, 2, …, k* and their inverses

M_i^{-1}, i = 1, 2, …, k* such that

$$M_i^{-1} = \begin{bmatrix} v_{i11} & v_{i21} & v_{1i1} \\ v_{i12} & v_{i22} & v_{i32} \\ v_{i13} & v_{i23} & v_{i33} \end{bmatrix}$$

Step 3: Compute the matrices of the interaction effect of the variables for the groups. These are

$$X_{iI} = \begin{bmatrix} x_{i11}^2 & x_{i11}x_{i21} & x_{i21}^2 \\ x_{i12}^2 & x_{i12}x_{i22} & x_{i22}^2 \\ \vdots & \vdots & \vdots \\ x_{i1N}^2 & x_{i1N}x_{i2N} & x_{i2N}^2 \end{bmatrix}$$

where i = 1, 2, …, k* and the vector of the interaction parameters obtained from f(**x**) is given by

$$\mathbf{g} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

The interaction vectors for the groups are given by $\mathbf{I}_i = M_i^{-1}X_i'X_{iI}\mathbf{g}$ and the matrices of mean square error for the groups are

$$\overline{M}_i = M_i^{-1} + \mathbf{I}_i\mathbf{I}_i' = \begin{bmatrix} \overline{v}_{i11} & \overline{v}_{i21} & \overline{v}_{i31} \\ \overline{v}_{i12} & \overline{v}_{i22} & \overline{v}_{i32} \\ \overline{v}_{i13} & \overline{v}_{i23} & \overline{v}_{i33} \end{bmatrix}$$

Step 4: Compute the optimal starting point, $\overline{\mathbf{x}}_1^*$ from

$$\overline{\mathbf{x}}_1^* = \sum_{m=1}^{N} u_m^*\mathbf{x}_m; \quad u_m^* > 0; \quad \sum_{m=1}^{N} u_m^* = 1, \quad u_m^* = \frac{a_m^{-1}}{\sum_{m=1}^{N} a_m^{-1}}, \quad a_m = \mathbf{x}_m'\mathbf{x}_m, \quad m = 1, 2, \ldots, N$$

Step 5: The matrices of coefficient of convex combinations of the matrices of mean square error are

$$H_i = \text{diag}\left\{\frac{\bar{v}_{i11}}{\sum \bar{v}_{i11}}, \frac{\bar{v}_{i22}}{\sum \bar{v}_{i22}}, \frac{\bar{v}_{i33}}{\sum \bar{v}_{i33}}\right\} = \text{diag}\{h_{i1}, h_{i2}, h_{i3}\}\ ,\ i = 1, 2, \ldots, k^*$$

By normalizing H_i such that $\sum H_i^* H_i^{*'} = I$, we have

$$H_i^* = \text{diag}\left\{\frac{h_{i1}}{\sqrt{\sum h_{i1}^2}}, \frac{h_{i2}}{\sqrt{\sum h_{i2}^2}}, \frac{h_{i3}}{\sqrt{\sum h_{i3}^2}}\right\}$$

The average information matrix is given by

$$M(\xi_N) = \sum_{i=1}^{k} H_i^* M_i H_i^{*'} = \begin{bmatrix} \bar{m}_{11} & \bar{m}_{21} & \bar{m}_{31} \\ \bar{m}_{12} & \bar{m}_{22} & \bar{m}_{32} \\ \bar{m}_{13} & \bar{m}_{23} & \bar{m}_{33} \end{bmatrix}$$

Step 6: From f(**x**), obtain the response vector

$$\mathbf{z} = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} \text{ where } z_0 = f(\bar{m}_{12} - \bar{m}_{13});\ z_1 = f(\bar{m}_{22} - \bar{m}_{23});\ z_2 = f(\bar{m}_{32} - \bar{m}_{33})$$

Hence, we define the direction vector

$$\mathbf{d} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} = M^{-1}(\xi_N)\mathbf{z}$$

and by normalizing **d** such that $\mathbf{d}^{*'}\mathbf{d}^{*} = 1$, we have

$$\mathbf{d}^* = \begin{bmatrix} d_1^* \\ d_2^* \end{bmatrix} = \begin{bmatrix} \dfrac{d_1}{\sqrt{d_1^2 + d_2^2}} \\ \dfrac{d_2}{\sqrt{d_1^2 + d_2^2}} \end{bmatrix}$$

Step 7: Obtain the step length, ρ_1^* from $\rho_1^* = \min_i \left\{\dfrac{\mathbf{c}_i'\bar{\mathbf{x}}_1^* - b_i}{\mathbf{c}_i'\mathbf{d}^*}\right\}$ where $\mathbf{c}_i'\mathbf{x} = b_i$, i = 1, 2, … , m is the i^{th} constraint of the quadratic programming problem.

Step 8: Make a move to the point $\mathbf{x}_2^* = \bar{\mathbf{x}}_1^* - \rho_1^*\mathbf{d}^*$

Step 9: Compute f($\mathbf{x}_2^*$) and f($\mathbf{x}_1^*$). Is $|f(\mathbf{x}_2^*) - f(\mathbf{x}_1^*)| \le \varepsilon$ where $\varepsilon = 0.0001$, then stop for the current solution is optimal, otherwise, replace $\bar{\mathbf{x}}_1^*$ by $\mathbf{x}_2^*$ and return to step 7. If the new step length, ρ_2^* is negligibly small, then an optimizer had been located at the first move.

4. A Numerical Example

An investor has a maximum of N10, 000.00 to invest by purchasing shares in Oceanic Bank and First City Monument Bank. Below is the historical data of prices per share in the banks for 25 days.
We are required to obtain optimal allocation of the investible funds for purchase of shares in the portfolio in order to minimize the total risk in the portfolio mix. From the data on table 3.1, the mean prices per share for First City Monument Bank and Oceanic Bank are 18.13 and 28.19 respectively.

Table 3.1: Price per share

Day	FCMB (Y_1)	Oceanic Bank (Y_2)	Day	FCMB (Y_1)	Oceanic Bank (Y_2)
1	17.6	29.89	14	18.5	26.95
2	17	29.61	15	18.78	26
3	17.55	28.95	16	18.4	27.3
4	17.9	27.95	17	18.74	28.86
5	17.5	28	18	18.74	30.09
6	17.7	28.61	19	19.1	30.6
7	17.74	28.6	20	18.71	29.6
8	17	26.49	21	18.9	28.6
9	16.8	25.99	22	18.75	29.01
10	16.99	25.95	23	18.53	28.98
11	18.5	27.3	24	18.5	28.55
12	18.8	27.17	25	18.49	28.75
13	18.1	26.99			
			$\sum Y_i$	453.32	704.79
			$\overline{Y}$	18.13	28.19

Source: The Nigerian Stock Exchange

Table 3.2: Mean deviation

Day	$(Y_1 - \overline{Y}_1)$	$(Y_2 - \overline{Y}_2)$	Day	$(Y_1 - \overline{Y}_1)$	$(Y_2 - \overline{Y}_2)$
1	-0.5328	1.6984	14	0.3672	-1.2416
2	-1.1328	1.4184	15	0.6472	-2.1916
3	-0.5828	0.7584	16	0.2672	-0.8916
4	-0.2328	-0.2416	17	0.6072	0.6684
5	-0.6328	-0.1916	18	0.6072	1.8984
6	-0.4328	0.4184	19	0.9672	2.4084
7	-0.3928	0.4084	20	0.5772	1.4084
8	-1.1328	-1.7016	21	0.7672	0.4084
9	-1.3328	-2.2016	22	0.6172	0.8184
10	-1.1428	-2.2416	23	0.3972	0.7884
11	0.3672	-0.8916	24	0.3672	0.3584
12	10.6672	-1.0216	25	0.3572	0.5584
13	-0.0328	-1.2016			

The expected return per share is the difference between the mean price of that share and its price on the 25th day. The investor assumes that his expected returns would be at least N100.00. Since his objective is to minimize his total risk, the problem involves obtaining optimal portfolio mix where the investment is done at the 25th day prices.

The share price deviations are obtained from table 3.1 as shown in table 3.2 while the variance – covariance matrix table for the share price are obtained from table 3.1 as shown in table 3.3.

From the table 3.3, the variance- covariance matrix is given by

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} 0.4863 & 0.3078 \\ 0.3078 & 1.7846 \end{pmatrix}$$

Hence, the model for minimizing the total risk of the portfolio is

$$\text{Min } f(x) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 0.4863 & 0.3078 \\ 0.3078 & 1.7846 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.4863x_1^2 + 0.6156x_1x_2 + 1.7846x_2^2$$

Subject to:

$$\begin{aligned} 18.49x_1 + 28.75x_2 &\le 10{,}000 \\ 0.3572x_1 + 0.5584x_2 &\ge 100 \\ x_1, x_2 &\ge 0 \end{aligned}$$

where x_1 and x_2 are respectively the number of shares purchased from First City Monument Bank and Oceanic Bank in the Portfolio.

Table 3.3: Variance- covariance matrix value

Day	$(Y_1-\bar{Y}_1)^2$	$(Y_1-\bar{Y}_1)(Y_2-\bar{Y}_2)$	$(Y_2-\bar{Y}_2)^2$	**Day**	$(Y_1-\bar{Y}_1)^2$	$(Y_1-\bar{Y}_1)(Y_2-\bar{Y}_2)$	$(Y_2-\bar{Y}_2)^2$
1	0.28387584	-0.90490752	2.88456256	14	0.13483584	-0.45591552	1.54157056
2	1.28323584	-1.60676352	2.01185856	15	0.41886784	-1.41840352	4.80311056
3	0.33965584	-0.44199552	0.57517056	16	0.07139584	-0.23823552	0.79495056
4	0.05419584	0.05624448	0.05837056	17	0.36869184	0.40585248	0.44675856
5	0.40043584	0.12124448	0.03671056	18	0.36869184	1.15270848	3.60392256
6	0.18731584	-0.18108352	0.17505856	19	0.93547584	2.32940448	5.80039056
7	0.15429184	-0.16041952	0.16679056	20	0.33315984	0.81292848	1.98359056
8	1.28323584	1.92757248	2.89544256	21	0.58859584	0.31332448	0.16679056
9	1.77635584	2.93429248	4.84704256	22	0.38093584	0.50511648	0.66977856
10	1.30599184	2.56170048	5.02477056	23	0.15776784	0.31315248	0.62157456
11	0.13483584	-0.32739552	0.79495056	24	0.13483584	0.13160448	0.12845056
12	0.44515584	-0.68161152	1.04366656	25	0.12759184	0.19946048	0.31181056
13	0.00107584	0.03941248	1.44384256				
				$\sum_{i=1}^{2}\sum_{j=1}^{2}(Y_i-\bar{Y}_i)(Y_j-\bar{Y}_j)$	11.670504	7.387288	42.830936
				σ_{ij}	0.486271	0.307803667	1.78462233

5. Test for Convexity

Since
$$\frac{\partial^2 f(x_j)}{\partial x_1^2}\frac{\partial^2 f(x_j)}{\partial x_2^2} - \left[\frac{\partial^2 f(x_j)}{\partial x_1 \partial x_2}\right]^2 = 3.0924 > 0$$

$$\frac{\partial^2 f(x_j)}{\partial x_1^2} = 0.9726 > 0 \quad \text{and} \quad \frac{\partial^2 f(x_j)}{\partial x_2^2} = 3.5692 > 0$$

f(**x**) is strictly a convex function and its global minimum point, **x*** is obtained by solving the above portfolio selection problem.

6. Solution to the portfolio selection problem by optimal designs approach

$$\text{Minimize } f(x) = 0.4863x_1^2 + 0.6156x_1x_2 + 1.7846x_2^2$$

Subject to:

$$18.49x_1 + 28.75x_2 \le 10{,}000$$
$$0.3572x_1 + 0.5584x_2 \ge 100$$
$$x_1, x_2 \ge 0$$

Let $\tilde{X}$ be the area defined by the constraint. Hence

$$\tilde{X} = \{x_1, x_2\}$$

Step 1: Select N support points such that $3k^* \le N \le 4k^*$ where $2 \le k^* \le 3$ is the number of partitioned groups desired. By choosing $k^* = 2$, we have $6 \le N \le 8$

Hence, by arbitrarily choosing 8 support points as long as they do not violate the constraints (within the feasible region), the initial design matrix is

$$x = \begin{bmatrix} 1 & 125 & 110 \\ 1 & 100 & 130 \\ 1 & 100 & 120 \\ 1 & 110 & 110 \\ 1 & 130 & 100 \\ 1 & 110 & 120 \\ 1 & 125 & 100 \\ 1 & 95 & 120 \end{bmatrix}$$

Step 2: Partition X into 2 groups such that

$$G_1 = \{x_1, x_2 \,; 100 \le x_1 \le 125, 110 \le x_2 \le 130\}$$
$$G_2 = \{x_1, x_2 \,; 95 \le x_1 \le 130, 100 \le x_2 \le 120\}$$

and the design matrices for the two groups are

$$X_1 = \begin{bmatrix} 1 & 125 & 110 \\ 1 & 100 & 130 \\ 1 & 100 & 120 \\ 1 & 110 & 110 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 & 130 & 100 \\ 1 & 110 & 120 \\ 1 & 125 & 100 \\ 1 & 95 & 120 \end{bmatrix}$$

The respective information matrices are

$$M_1 = X_1^{'}X_1 = \begin{bmatrix} 4 & 435 & 470 \\ 435 & 47725 & 50850 \\ 470 & 50850 & 55500 \end{bmatrix} \text{ and } \quad M_2 = X_2^{'}X_2 = \begin{bmatrix} 4 & 460 & 440 \\ 460 & 53650 & 50100 \\ 440 & 50100 & 48800 \end{bmatrix}$$

Step 3: The matrices of the interaction effect of the variables are

$$X_{1I} = \begin{bmatrix} 15625 & 13750 & 12100 \\ 10000 & 13000 & 16900 \\ 10000 & 12000 & 14400 \\ 12100 & 12100 & 12100 \end{bmatrix} \text{ and } \quad X_{2I} = \begin{bmatrix} 16900 & 13000 & 10000 \\ 12100 & 13200 & 14400 \\ 15625 & 125000 & 10000 \\ 9025 & 14400 & 14400 \end{bmatrix}$$

and the vector of the interaction parameters obtained from f(x) is given by

$$\mathbf{g} = \begin{bmatrix} 0.4863 \\ 0.6156 \\ 1.7846 \end{bmatrix}$$

The interaction vectors for the groups are

$$\mathbf{I_1} = M_1^{-1} X_1' X_{1I} \mathbf{g} = \begin{bmatrix} -40534 \\ 186 \\ 499 \end{bmatrix}$$

and

$$\mathbf{I_2} = M_2^{-1} X_2' X_{2I} \mathbf{g} = \begin{bmatrix} -34541 \\ 175 \\ 459 \end{bmatrix}$$

The matrices of mean square error for the groups are respectively

$$\overline{M}_1 = M_1^{-1} + I_1 I_1' = \begin{bmatrix} 1643005496.62 & -7539324.31 & 20226464.31 \\ -7539324.31 & 34596.00 & 94311.00 \\ 20226464.31 & 94311.00 & 249001.00 \end{bmatrix}$$

$$\overline{M}_2 = M_2^{-1} + I_2 I_2' = \begin{bmatrix} 1193081221.55 & 6044672.98 & 15854316.20 \\ 6044672.98 & 30625.01 & 80325.01 \\ 15854316.20 & 80325.01 & 210681.02 \end{bmatrix}$$

Step 4: Obtain the optimal starting point

$$\overline{x}_1^* = \sum_{m=1}^{N} u_m^* x_m;\ u_m^* > 0;\ \sum_{m=1}^{N} u_m^* = 1,\ u_1^* = \frac{a_m^{-1}}{\sum_{m=1}^{N} a_m^{-1}},\quad a_m = x_m' x_m,\ m = 1, 2, ..., N$$

Now,

$$a_1 = \mathbf{x_1'x_1} = [1 \quad 125 \quad 110] \begin{bmatrix} 1 \\ 125 \\ 110 \end{bmatrix} = 27726,\ a_1^{-1} = 0.00003607 \qquad a_2 = \mathbf{x_2'x_2} = [1 \quad 100 \quad 130] \begin{bmatrix} 1 \\ 100 \\ 130 \end{bmatrix} = 26901,\ a_2^{-1} = 0.00003717$$

$$a_3 = \mathbf{x_3'x_3} = [1 \quad 100 \quad 120] \begin{bmatrix} 1 \\ 100 \\ 120 \end{bmatrix} = 24401,\ a_3^{-1} = 0.00004098 \qquad a_4 = \mathbf{x_4'x_4} = [1 \quad 110 \quad 110] \begin{bmatrix} 1 \\ 110 \\ 110 \end{bmatrix} = 24201,\ a_4^{-1} = 0.00004132$$

$$a_5 = \mathbf{x_5'x_5} = [1 \quad 130 \quad 100] \begin{bmatrix} 1 \\ 130 \\ 100 \end{bmatrix} = 26901,\ a_5^{-1} = 0.00003717 \qquad a_6 = \mathbf{x_6'x_6} = [1 \quad 110 \quad 120] \begin{bmatrix} 1 \\ 110 \\ 120 \end{bmatrix} = 26501,\ a_6^{-1} = 0.00003773$$

$$a_7 = \mathbf{x_7'x_7} = [1 \quad 125 \quad 100] \begin{bmatrix} 1 \\ 125 \\ 100 \end{bmatrix} = 25626,\ a_7^{-1} = 0.00003902 \qquad a_8 = \mathbf{x_8'x_8} = [1 \quad 95 \quad 125] \begin{bmatrix} 1 \\ 95 \\ 125 \end{bmatrix} = 23426,\ a_8^{-1} = 0.00004269$$

$$\sum_{m=1}^{8} a_m^{-1} = 0.00003607 + 0.00003717 + 0.00004098 + 0.00004132 + 0.00003717 + 0.00003773$$
$$+ 0.00003902 + 0.00004269 = 0.0003122$$

Since

$$u_1^* = \frac{a_m^{-1}}{\sum_{m=1}^{N} a_m^{-1}},\ m = 1, 2, ..., N$$

$$u_1^* = \frac{0.00003607}{0.0003122} = 0.1155\,,\ \ u_2^* = \frac{0.00003717}{0.0003122} = 0.0119$$
$$u_3^* = \frac{0.00004098}{0.0003122} = 0.1313\quad u_4^* = \frac{0.00004132}{0.0003122} = 0.1324$$
$$u_5^* = \frac{0.00003717}{0.0003122} = 0.1191,\ \ u_6^* = \frac{0.00003773}{0.0003122} = 0.1209$$

$$u_7^* = \frac{0.00003902}{0.0003122} = 0.1250\ ,\ \ u_8^* = \frac{0.00004269}{0.0003122} = 0.1367$$

Hence, the optimal starting point is $\bar{\mathbf{x}}_1^* = \sum_{m=1}^{8} u_m^* x_m$

$$\bar{x}_1^* = 0.1155\begin{bmatrix}1\\125\\110\end{bmatrix} + 0.0119\begin{bmatrix}1\\100\\130\end{bmatrix} + 0.1313\begin{bmatrix}1\\100\\120\end{bmatrix} + 0.1324\begin{bmatrix}1\\110\\110\end{bmatrix}$$
$$+\ 0.1191\begin{bmatrix}1\\130\\100\end{bmatrix} + 0.1209\begin{bmatrix}1\\110\\120\end{bmatrix} + 0.1250\begin{bmatrix}1\\125\\100\end{bmatrix}$$
$$+\ 0.1367\begin{bmatrix}1\\95\\120\end{bmatrix} = \begin{bmatrix}\underline{1.0000}\\111.4350\\113.8299\end{bmatrix}$$

Step 5: Obtain the matrices of coefficients of convex combinations from $\overline{M}_1$ and $\overline{M}_2$ as follows:

$$H_1 = \mathbf{diag}\left\{\frac{1193081221.55}{1193081221.55 + 1643005496.62},\quad \frac{30625.01}{30625.01 + 34596.01},\quad \frac{210681.02}{210681.02 + 249001.02}\right\} = \text{diag}\{0.4207,\ 0.4696,\ 0.4583\}$$

$$H_2 = I - H_1 = \mathbf{diag}\{0.5793,\ 0.5304, 0.5417\}$$

and by normalizing H_1 and H_2 such that $H_1^* H_1^{*'} + H_2^* H_2^* = 1$, we have

$$H_1^* = \mathbf{diag}\left\{\frac{0.4207}{\sqrt{0.4207^2 + 0.5793^2}}, \frac{0.4696}{\sqrt{0.4696^2 + 0.5304^2}}, \frac{0.4583}{\sqrt{0.4583^2 + 0.5417^2}}\right\}$$

$$= \mathbf{diag}\{0.5876, 0.6629, 0.6459\}$$

$$H_2^* = \mathbf{diag}\left\{\frac{0.5793}{\sqrt{0.4207^2 + 0.5793^2}}, \frac{0.5304}{\sqrt{0.4696^2 + 0.5304^2}}, \frac{0.5417}{\sqrt{0.4583^2 + 0.5417^2}}\right\}$$

$$= \mathbf{diag}\{0.8091, 0.7487, 0.7634\}$$

The average information matrix is given by

$$M(\xi_N) = H_1^* X_1^{'} X_1 H_1^{*'} + H_2^* X_2^{'} X_2 H_2^{*'}$$

$$= \begin{bmatrix} 4 & 448 & 450 \\ 448 & 51046 & 50408 \\ 450 & 50408 & 51595 \end{bmatrix}$$

Step 6: From $f(x_1, x_2)$, obtain the response vector $\mathbf{z} = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix}$

$$z_0 = f(448, 450) = 0.4863(448)^2 + 0.615644(448)(450)$$
$$+1.7846(450)^2 = 583090$$
$$z_1 = f(51046,\ 50408) = 0.4863(510\,46)^2 + 0.615644(5\,1046)(5040\,8)$$
$$+1.7846(504\,08)^2 = 7385800000$$
$$z_2 = f(50408, 51595) = 0.4863(50408)^2 + 0.615644(50408)(51595)$$
$$+1.7846(51595)^2 = 7587400000$$

Therefore,

$$\mathbf{z} = \begin{bmatrix} 583090 \\ 7385800000 \\ 7587400000 \end{bmatrix}$$

Here, we define the direction vector

$$\mathbf{d} = \begin{bmatrix} \underline{d_0} \\ d_1 \\ d_2 \end{bmatrix} = M^{-1}(\xi_N)\mathbf{z} = \begin{bmatrix} \underline{-1936900000} \\ 8900000 \\ 8300000 \end{bmatrix}$$

and by normalizing **d** such that $\mathbf{d}^{*'}\mathbf{d}^* = \mathbf{1}$, we have

$$\mathbf{d}^* = \begin{bmatrix} d_1^* \\ d_2^* \end{bmatrix} = \begin{bmatrix} \dfrac{8900000}{\sqrt{8900000^2 + 8300000^2}} \\ \dfrac{8300000}{\sqrt{8900000^2 + 8300000^2}} \end{bmatrix} = \begin{bmatrix} 0.7313 \\ 0.6820 \end{bmatrix}$$

Step 7: Obtain the step length, ρ_i^* from

$$\rho_i^* = \min_i \left\{ \frac{c_i^{'}\bar{\mathbf{x}}_1^* - b_i}{c_i^{'}d^*} \right\}$$

where $c_i^{'}x = b_i$, $i = 1, 2, ..., m$ is the ith constraint of the portfolio selection problem.

For $c_1 = \begin{bmatrix} 18.49 \\ 28.75 \end{bmatrix}$ and $b_1 = 10000$, we have $\rho_1^* = \left\{ \dfrac{[18.49 \quad 28.75]\begin{bmatrix} 111.4350 \\ 113.8299 \end{bmatrix} - 10000}{[18.49 \quad 28.75]\begin{bmatrix} 0.7313 \\ 0.6820 \end{bmatrix}} \right\} = -140.8659$

For $c_2 = \begin{bmatrix} 0.3572 \\ 0.5584 \end{bmatrix}$ and $b_2 = 100$, we have $\rho_2^* = \left\{ \dfrac{[0.3572 \quad 0.5584]\begin{bmatrix} 111.4350 \\ 113.8299 \end{bmatrix} - 100}{[0.3572 \quad 0.5584]\begin{bmatrix} 0.7313 \\ 0.6820 \end{bmatrix}} \right\} = 5.2443$

Step 8: Make a move to the point

$$\mathbf{x}_2^* = \bar{\mathbf{x}}_1^* - \rho_1^* d^* = \begin{bmatrix} 111.4350 \\ 113.8299 \end{bmatrix} - [-140.8659]\begin{bmatrix} 0.7313 \\ 0.6820 \end{bmatrix} = \begin{bmatrix} 214.4242 \\ 209.9040 \end{bmatrix},$$

since $\rho_1^* = -140.8659$ is the minimum step length.

Step 9

$$\begin{aligned} f(\mathbf{x}_2^*) &= 0.4863(214.4242)^2 + 0.6156(214.4242)(209.9040) \\ &\quad + 1.7846(209.9040)^2 = 128700 \\ f(\bar{\mathbf{x}}_1^*) &= 0.4863(111.4350)^2 + 0.6156(111.4350)(113.8299) \\ &\quad + 1.7846(113.8299)^2 = 36971 \end{aligned}$$

$$\text{since } \left| f(\mathbf{x}_2^*) - f(\bar{\mathbf{x}}_1^*) \right| = |128700 - 36971| = 91729$$

Make a second move by replacing $\bar{\mathbf{x}}_1^* = \begin{bmatrix} 111.4350 \\ 113.8299 \end{bmatrix}$ by $\mathbf{x}_2^* = \begin{bmatrix} 214.4242 \\ 209.9040 \end{bmatrix}$

The new step length is obtained as follows: $\rho_3^* = \left\{ \dfrac{[18.49 \quad 28.75]\begin{bmatrix} 214.4242 \\ 209.9040 \end{bmatrix} - 100}{[18.49 \quad 28.75]\begin{bmatrix} 0.7313 \\ 0.6820 \end{bmatrix}} \right\} = -0.000125$

Since the new step length is negligible, the optimal solution was obtained at the first move and hence,

$$\mathbf{x}_2^* = \begin{bmatrix} 214.4242 \\ 209.9040 \end{bmatrix} \text{ and } f(\mathbf{x}_2^*) = 1\,28700$$

The portfolio selection problem which is a minimization of portfolio variance was solved using modified super convergent line series algorithm which gave

$$x_1 = 214$$
$$x_2 = 210$$

as the number of shares to be purchased from Oceanic Bank and First City Monument Bank respectively in order to obtain a minimum risk or minimum variance.

7. Summary and Conclusion

In this paper, we assumed that the portfolio has already been selected by the investor from a list of available investments. Using historical data prices (25 days) of stocks from First City Monument Bank and Oceanic Bank, we showed how optimal allocations of investible funds could be made to each Bank's stocks by minimizing the portfolio variance thereby minimizing the total risk using optimal designs approach.

The approach adopted in obtaining optimal solution is recommended for use by potential investors as a way out of business collapse.

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